Linearised Optimal Power Flow Problem Solution using Dantzig - Wolfe decomposition

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***Abstract*—The optimal power flow problem can be used to determine the generation dispatch in electrical grids taking into account circuit physics and technical constraints. This paper uses an old-fashioned mathematical decomposition technique, the Dantzig-Wolfe decomposition, to solve such problems and compares the results to state-of-the-art commercial solvers.**
The linearised 'DC' OPF model is reformulated to create a block angular structure of the problem and is optimized through a column generation algorithm based on the simplex method. Columns of variables are generated for each iteration of the optimization process, providing a solution where selected vertices from the feasible sets are assigned a weight and combined to get the optimal value.
The results of three test cases consist of an useful starting point for further works with larger test systems and MILP problems such as the transmission network expansion planning problems.

I. Introduction

A. Context and motivation

The Optimal Power Flow (OPF) problem allows to optimize control variables in power systems with respect to a defined objective function [1]. The aim of the OPF problem is to minimize the objective function by ensuring that all the constraints in the formulations are satisfied. The power flow equations, technical equipment limits and operational limits build the constraints of the problem [2]. The OPF problem forms the basis for other more complex power system optimisation problems such as the security-constrained OPF or the transmission expansion planning (TNEP) problem, which have high practical value for system operators. Hence, the efficient and robust solution to the OPF problem also opens the possibility for a more efficient solution to its more more complex variants.
This paper provides exploratory work on the solution to the OPF problem using Dantzig-Wolfe reformulation (DW) [3]. The linearised 'DC' OPF approximation [4] is used for the application of the DW reformulation. In order to test the developed method, three examples of small power systems are taken from [5] and the computational performance is validated against existing optimal power flow tools, e.g., PowerModels.jl [6] using the Gurobi solver [7].

B. Dantzig-Wolfe decomposition in the literature

In literature, decomposition techniques such as the Benders decomposition [8], and its nested extension [9], have been used as solution techniques for large-scale mixed-integer linear programming (MILP) problems. Nevertheless, there is still a lack of literature on the application of the DW decomposition, combined with a Column Generation (CG) algorithm [10], to OPF problems. In fact, in the current literature, the DW reformulation coupled with CG methods is mainly used in the context of operations research, such as in [11]. Regarding its application in the context of power system optimisation, the DW decomposition is restricted to demand response applications [12], peak minimization for demand response applications [13] and optimal VAr planning problems in multi-area electric power systems [14].

The main contributions of this paper are (i) the reformulation of the linearised optimal power flow model to create a block structure, and (ii) the development of a column generation method for the application of the simplex method to solve the OPF problem. The paper is structured as follows. Section II provides the reformulation of the problem to create a block structure and outlines the column generation method. Section III applies the methodology to a number of test cases and analyses the convergence behaviour. Finally, section IV summarizes the conclusions and provides future research directions.

II. Methodology

A. DC OPF problem

The original problem consists of the DC OPF formulation as described in (1) - (6) [15], where the nonlinear constraints have been linearised and several assumptions have been taken on the system characteristics [16].

$$min(C) = \sum_{g}^{G} P_g \cdot C_g + \sum_{lfe}^{L} P_{lfe} \cdot C_l \quad \forall g \in G, lfe \in L \quad (1)$$

s.t.

$$\sum_{lfe}^{L} P_{lfe} + \sum_{g}^{G} P_g = \sum_{m}^{M} P_m \quad \forall n \in N \tag{2}$$

$$P_{lfe} - b_{lfe}(\theta_f - \theta_e) = 0 \quad \forall lfe \in L \tag{3}$$

$$P_{lfe}^{min} \leq P_{lfe} \leq P_{lfe}^{max} \quad \forall lfe \in L \tag{4}$$

$$0 \leq P_g \leq P_g^{max} \quad \forall g \in G \tag{5}$$

$$\theta_{ref} = 0 \tag{6}$$

The objective function (1) minimizes the total generation cost. The term C_l has a value of zero for each branch.
Constraint (2) describes the nodal power balance (or Kirchoff's current law) for each bus $n \in N$. The terms in (2) refer to the branches $lfe \in L$, generators $g \in G$ and loads $m \in M$. The f and in e in lfe respectively refer to the from and to bus of the branch.
Equation (3) represents Kirchoff's voltage and Ohm's laws respectively for the branches $lfe \in L$, while (4) and (5) are the active power limits for namely the existing branches $lfe \in L$ and generators $g \in G$. Lastly, (6) provides the voltage angle reference in the system.

B. Dantzig-Wolfe decomposition, Column Generation and Simplex method

The main idea behind the DW decomposition is to express the polyhedron of the original problem, i.e. the feasible set, as a combination of extreme points $Q = x_1, ..., x_Q$ and extreme rays $R = r_1, ..., r_R$. Therefore, each feasible set is represented as a multitude of vertices in the solution space linked by lines, or rays. The combination of these object define the boundaries of the possible solutions for the variables in the formulation. Particularly, the vertices are constrained by the minimum and maximum value a variable can take in the optimization process.
In order to simplify the formulation without losing accuracy, only the extreme points (or vertices) Q are taken into account in this paper. Since the feasible sets of the test cases are fully bounded, the influence of the extreme rays R is negligible.

$$z^* = \min_{X_g, X_p} \sum_{i}^{X_g} (\sum_{g}^{G} C_g \cdot x_{g,i}) \lambda_{sp1,i} + \sum_{j}^{X_l} (\sum_{l}^{S} C_l \cdot x_{l,j}) \lambda_{sp2,j} \tag{7}$$

s.t.

$$\sum_{i}^{X_g} (\sum_{g}^{G} A_1 \cdot x_{g,i}) \lambda_{sp1,i} + \sum_{j}^{X_l} (\sum_{l}^{S} A_2 \cdot x_{l,j}) \lambda_{sp2,j} \leq b_m \; (y_n) \tag{8}$$

$$\sum_{i}^{X_g} \lambda_{sp1,i} = 1 \; (\alpha_1) \tag{9}$$

$$\sum_{j}^{X_l} \lambda_{sp2,j} = 1 \; (\alpha_2) \tag{10}$$

The OPF problem after DW decomposition is based on two subproblems respectively linked to the set of generators and branches in the power system. Equations (7)-(10) define the Master Problem (MP). It computes the objective cost z^* by taking into account all the vertices in the two polyhedrons X_g and X_l. Each vertex is assigned a variable $\lambda_{sp1,i}$ or $\lambda_{sp2,j}$ which corresponds to its weight in the final solution.
The linking or complicating constraints are expressed by equation (8), where the two matrices A_1 and A_2 are related to the subproblems. They contain the coefficients for each variable in the subproblem based on their presence in the linking constraints. In this reformulation, the linking or complicating constraints consist of the nodal balance and power flow equations defined previously in the text in equations (2) and (3).
Furthermore, the b_m vector correspond to the load demand for each node in the power system. The number of linking constraints is therefore the number of nodes N in the power system. Each element of this set of complicating constraints has a dual y_n being computed.
Finally, as defined by equations (9) and (10), the sum of the weights for each subproblem must be equal to one to ensure the convexity of the problem. The duals α_1 and α_2 of these convexity constraints are used during each iteration of the CG method.
The number of vertices in the feasible set scales with a factor of 2^n, where n is the number of variables in the system, e.g. the active power set points from a set of n generators. It becomes clear that the problem becomes numerically intractable for higher values of n.
For this reason, the CG method as depicted in Fig. 1 is used.

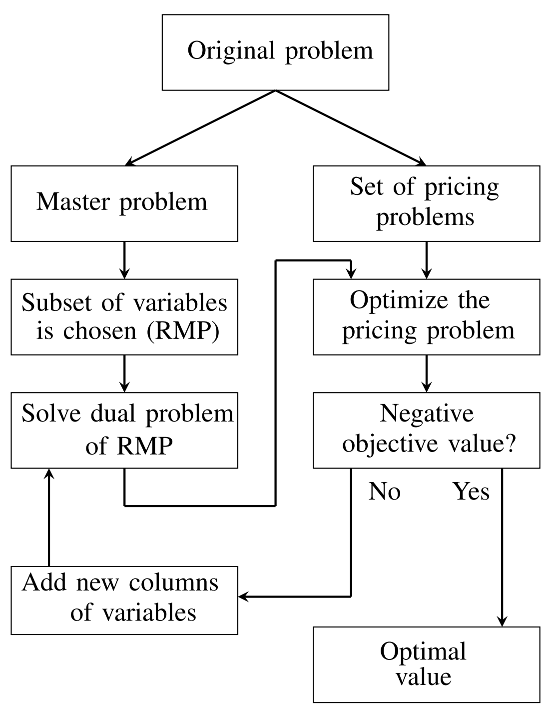

Fig. 1. Column Generation method flow chart.

The MP is reduced to a Restricted Master Problem (RMP) where only a small subset of vertices is chosen and updated for each iteration of the CG algorithm. The RMP correlates the pricing problems and verifies that the global constraints are satisfied [17]. The CG algorithm determines the columns of variables to be added by combining the objective value of the subproblems, computed following equation (11), with the simplex method [18]. The simplex tableau used in this paper is provided in Table I.

$$z^*_{spi} = min(c^T_{spi} - y_n \cdot A_i)x_{spi} \tag{11}$$

The elements $y_1, ..., y_N$ and α_1, α_2 are the duals of respectively the linking constraints and convexity constraints, as indicated previously in the text. The Basic Variables (BV) are the weights λ_{sp} assigned to each vertex and their values are computed by $B_{inv} * b_m$. B_{inv} is the matrix indicating the initial values for each variable at each iteration, obtained by multiplying the A_i matrix by the incoming vertex. Initially, B_{inv} is an identity matrix, since the starting point for both subproblems is a vertex whose coefficients are only zeros.

TABLE I
SIMPLEX METHOD TABLE

y_1	...	y_N	α_1	α_2	z	BV
						λ_1
						...
	B_{inv}				B_{inv}*b	
						λ_n
						λ_{n+1}
						λ_{n+2}

For this reason, b_m assumes the values of the slack variables s_n needed to start the algorithm and is defined as the vector of the basic variables. Equation (8) can therefore be rewritten as (12), where n is the number of linking equations.

$$\sum_i^{X_g}(\sum_g^G A_1 \cdot x_{g,i})\lambda_{sp1,i} + \sum_j^{X_l}(\sum_l^S A_2 \cdot x_{l,j})\lambda_{sp2,j} + s_n = b_m \tag{12}$$

A new column enters the table for each iteration, depending on the highest objective value among the subproblems. Its position is based on a ratio test between the basic variable values and the column entering the algorithm, with all the negative basic variables and zeros in the column coefficients not taken into account.

If none of the subproblems leads to a reduced cost, i.e. their objective values are lower or equal to 0, the CG algorithm stops.

The objective value z^* of the MP is updated for each iteration. In the case of a minimization with two subproblems, the lower bound (LB) is computed as $LB = z^* - z^*_{sp,1} - z^*_{sp,2}$. For a maximisation, the upper bound (UB) is defined vice versa as $UB = z^* + z^*_{sp,1} + z^*_{sp,2}$. When the LB/UB and z^* converge to the same result, within a certain tolerance, the optimum is found.

Fig. 2 summarizes the interrelations and exchange of values between the MP and the related subproblems for the DW decomposition [13].

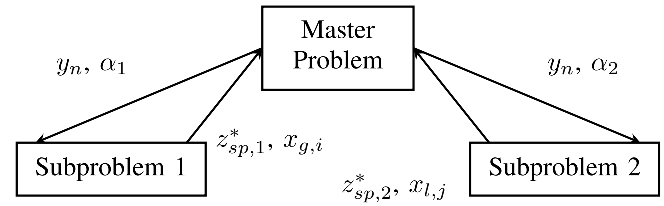

Fig. 2. Exchange of variables between the MP and the related subproblems.

C. DC OPF Problem with Dantzig-Wolfe decomposition

In order to apply the DW decomposition in combination with a CG method, a block-angular structure of the optimisation problem is necessary. In this sense, Fig. 3 shows the block angular structure of the DC OPF problem formulated in equations (1)-(6), where the equations are assigned to the different parts of the structure.

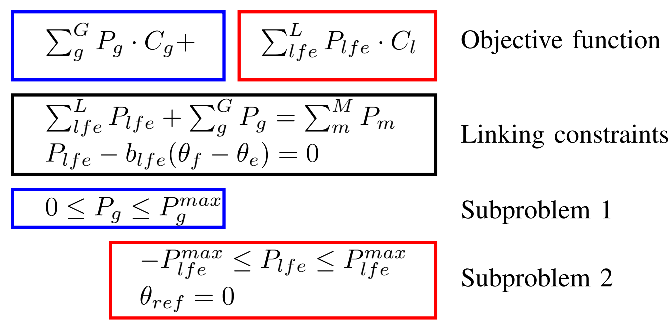

Fig. 3. Block angular structure of the DC OPF problem.

The DC OPF-DW problem is divided into a MP and two subproblems: the first one, SP_1, is linked to the power generation in the system. In SP_1, there are G variables where G is the number of generators. In SP_2, the number of variables corresponds to $S = L + N$, where L and N are the numbers of branches and nodes, respectively, as the nodal voltage angles are linked to the power flows through the lines in (3).

Therefore, the matrices of coefficients for each variable in problem A_1 and A_2 refer to the numner S of linking constraints defined by equations (2) and (3). For this reason, the matrices have both S rows. The number of columns corresponds instead to the number of variables for each subproblem, namely G for A_1 and S for A_2.

The formulation of the MP is expressed in previous general equations (7)-(10) from section II-B, while the subproblems of this particular formulation are defined in the following two subsections.

1) Subproblem 1: The objective value z^*_{sp1} and constraints of SP_1 are defined as:

$$z^*_{sp1} = max\,(\sum_g^G C_g^T \cdot x_{g,i} - \sum_i^S(y_n^T \cdot A_1)x_{g,i} - \alpha_1) \tag{13}$$

s.t.

$$x_{g,i} \leq P_g^{max} \quad \forall g \in G \tag{14}$$

$$x_{g,i} \geq 0 \quad \forall g \in G \tag{15}$$

Each $x_{g,i}$ belonging to the subproblem SP_1 is constrained by the minimum and maximum generation capacity P_g^{min} and P_g^{max}, respectively.

2) Subproblem 2: In subproblem SP_2, the variables are limited by the power rating P_{lfe}^{min}, P_{lfe}^{max} of the branches in (17) and by the voltage angle bounds in (18). Equation (19) provides the reference for the voltage angle.

$$z_{sp2}^* = max \; (\sum_j^S C_l^T \cdot x_{l,j} - \sum_j^S (y_j^T \cdot A_2) x_{l,j} - \alpha_2) \tag{16}$$

s.t.

$$P_{lfe}^{min} \leq x_{l,j} \leq P_{lfe}^{max} \quad \forall j \in 1 : L \tag{17}$$

$$\theta_n^{min} \leq x_{l,j} \leq \theta_n^{max} \quad \forall j \in L+1 : S \tag{18}$$

$$x_{l,j} = 0 \quad \forall j = L+1 \tag{19}$$

III. Results

In this section, the results of the three test cases are analysed. Firstly, the DW & CG method is tested against PowerModels [6] using Gurobi solver v9.1.2 [7]. Secondly, the relation between the objective value and the lower bound of each investigated test case is shown. The DW & CG method is implemented in Julia / JuMP [19].

A. Dantzig-Wolfe decomposition versus Gurobi

The comparison between the two methods is shown in table II. By using Gurobi with the barrier optimization method, the optimization process takes both less time and iterations to reach the objective value compared to the DW & CG algorithm with a tolerance of 10^{-5}.
In addition, it is noticed that the number of iterations and the computational time is proportionally dependent to the number of buses in the test case.

TABLE II
Computational time and number of iterations needed to reach optimality for the three test cases with Gurobi and the DW&CG method.

	Gurobi [s]	Iterations	DW & CG [s]	Iterations
Case 3	0.004722	1	0.09355	7
Case 5	0.006595	2	0.180879	12
Case 14	0.010442	9	0.487918	28

B. Analysis of the three test cases

The behaviour of both the computed objective values and lower bounds throughout each iteration of the method is shown in Figures 4, 5 and 6.
For all cases, the relative objective value has a steady behaviour while the lower bound oscillates between different negative values before becoming positive and reaching optimality. This can be explained by the fact that the starting point of the method is a vector of zeros. For this reason, the method takes some iterations and a wider range of values to find the right combination of power generation and branch flows. By providing a warm start value, the oscillations can be significantly smoothened out [20], but this technique is beyond the scope of this paper.

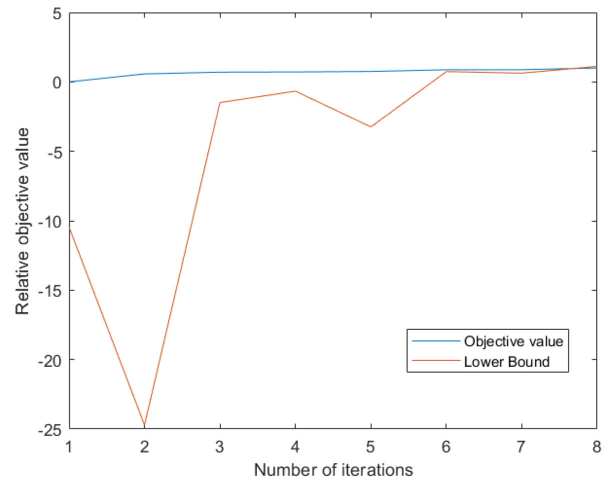

Fig. 4. Comparison between the lower bound and the objective value for each iteration of the Column Generation algorithm in the 3-bus case.

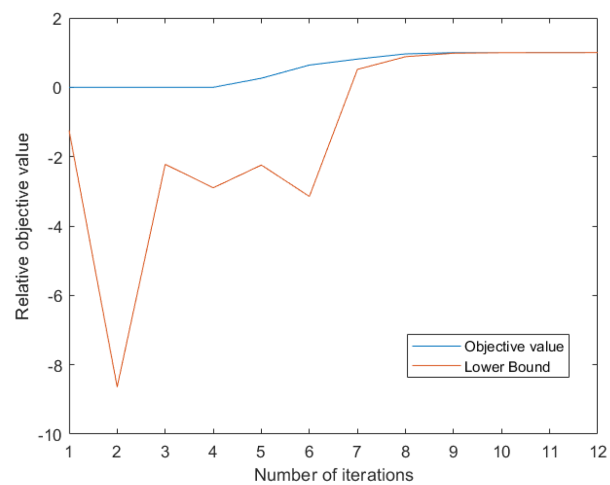

Fig. 5. Comparison between the lower bound and the objective value for each iteration of the Column Generation algorithm in the 5-bus case.

Fig. 7 shows the main feature of the DW decomposition. The three axes represent the P_g^{min} minimum and P_g^{max} maximum active power set points of the three generators in the 3-bus case. As the number of vertices in the feasible set scales with 2^n, there are 8 available vertices for this example. The final power generation is the combination of three vertices, weighted by their $\lambda_{sp1,i}$. As anticipated by the convexity constraint (9), the three $\lambda_{sp1,i}$ variables sum up to 1.
The results for the 3-bus case are summarised in Table III, where the vertices of Fig. 7 are multiplied by their $\lambda_{sp1,i}$. By summing each vertex-$\lambda_{sp1,i}$ couple, the final active power for

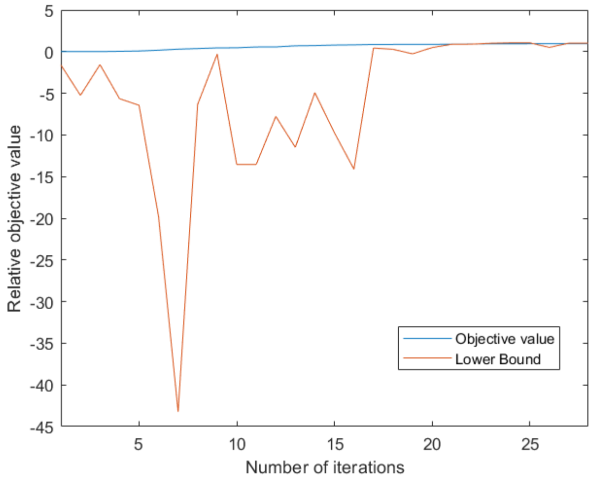

Fig. 6. Comparison between the lower bound and the objective value for each iteration of the Column Generation algorithm in the 14-bus case.

each generator is computed.

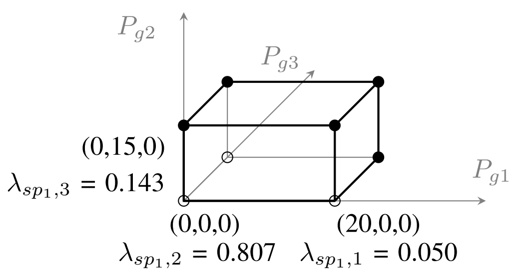

Fig. 7. Representation of the feasible set in the 3-bus case. The assigned vertices and respective weigths for the active power in the set of three generators are shown and visually represented by the empty circles.

TABLE III
ACTIVE POWER GENERATION FOR EACH VERTEX AND ASSOCIATED $\lambda_{sp1,n}$ WEIGHT

Basic variable		Vertex	Generation [pu]
$\lambda_{sp1,1}$	0.05	(20, 0, 0)	(1.0, 0, 0)
$\lambda_{sp1,2}$	0.807	(0, 0, 0)	(0, 0, 0)
$\lambda_{sp1,3}$	0.143	(0, 15, 0)	(0, 2.145, 0)
Total generation			(1.0, 2.145, 0)

IV. Conclusion and Future Work

The paper presents an innovative method of solving linear OPF problems by applying the Dantzig-Wolfe decomposition and the Column Generation method. Although the proof-of-concept implementation provided in this paper results in higher computation times, for larger test systems, computational benefits can be expected, especially through the provision of warm-start values.

Future work will consist of applying this method to larger test cases for detailed analysis. In addition, the technique can be extended to Transmission Network Expansion Planning (TNEP) problems, where the main source of high computational times is the presence of binary variables. Relaxing them for the master problem and applying the DW & CG method to the updated TNEP problem by means of only adding binary variables for most the relevant variables/columns can lead to improved performance in terms of computational time.

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